Positive Stable Laws of Stability 2^{-n}

Angelos Dassios[*]
London School of Economics

Yan Qu[†]
London School of Economics

Hongbiao Zhao[‡]
Shanghai University of Finance and Economics

29th May 2019

Abstract

We derive an integral representation for the density function of positive stable distribution with stability index 2^{-n}, $n \in \mathbb{N}^{+}$. This result immediately reveals an elegant decomposition for this distribution into simple inverse gamma distributions in a recursive structure. It also leads to a simple procedure for exact simulation.

Keywords: Stable distributions; Stable laws; Heavy tails; Density functions; Monte Carlo simulation

Mathematics Subject Classification (2010): Primary: 60E07; Secondary: 60G51 · 60G52 · 65C10

1 Introduction

Lévy (1925) first introduced the family of stable distributions which offers a great alternative to the Gaussian distributions. Tremendous applications of stable distributions and their various extensions in finance, economics and many other fields can be found in the literature, see Mandelbrot (1963); Fama (1965); Samoradnitsky and Taqqu (1994); McCulloch (1996); Cont et al. (1997); Uchaikin and Zolotarev (1999); Jondeau et al. (2007); Cizek et al. (2011).

In this note, we concentrate on an interesting special family, the *stable distribution of stability* 2^{-n} for $n \in \mathbb{N}^{+}$:

[*]Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Email: a.dassios@lse.ac.uk

[†]Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Email: y.qu3@lse.ac.uk

[‡]*Corresponding author*, School of Statistics and Management, Shanghai University of Finance and Economics, No. 777 Guoding Road, Shanghai 200433, China. Email: h.zhao1@lse.ac.uk

Definition 1.1 (Stable Distribution of Stability 2^{-n})**.** *Stable distribution of stability 2^{-n}, denoted by $\mathcal{S}\left(2^{-n},\kappa\right)$, is a one-sided positive stable distribution with the Laplace transform*

$$\mathbb{E}\left[e^{-u\mathcal{S}\left(2^{-n},\kappa\right)}\right]=\exp\left(-\kappa u^{2^{-n}}\right),\qquad u,\kappa\in\mathbb{R}^{+},\qquad n\in\mathbb{N}^{+},\tag{1.1}$$

where $2^{-n}\in(0,1)$ is the stability index and κ is the scale parameter.

As the name "stable" implies in the stable distribution, *stability index* is the most crucial parameter, since this fundamentally determines its distributional property of *stable laws* (Borak et al., 2011, p.23). It has been commonly recognised that, the density functions for general specifications on the stability index are hard to be obtained analytically, and this poses the greatest challenge to the further study of their distributional properties and statistical inference.

In the literature, Brown and Tukey (1946) and Mitra (1981, 1982, 1983) obtained a very elegant representation:

$$\mathcal{S}(2^{-n},1)\overset{\mathcal{D}}{=}\begin{cases}Y_1, & \text{if}\qquad n=1,\\ Y_1\left(Y_2b_2^2\right)^2\left(Y_3b_3^2\right)^4\ldots\left(Y_nb_n^2\right)^{2^n-1}, & \text{if}\qquad n\geq 2,\end{cases}\tag{1.2}$$

where

$$b_i:=\frac{1}{\sqrt{2\sec\left(\frac{1}{2^i}\pi\right)\cos\left(\frac{1}{2^{i+1}}\pi\right)}},\qquad i=2,3,...,n,$$

and $\{Y_i\}_{i=1,2,\ldots,n}$ are *i.i.d.* random variables with $Y_i\sim\mathsf{InvGamma}\left(\frac{1}{2},\ \frac{1}{2}\right)$, i.e. an inverse gamma distribution with the shape parameter $\frac{1}{2}$ and the rate parameter $\frac{1}{2}$. Whereas in this note, we provide some representations particularly for $\mathcal{S}\left(2^{-n},\kappa\right)$ of Definition 1.1, which imply very efficient simulation algorithms.

2 Integral Representations and Distributional Decompositions

We derive a new multiple integral representation for the density function of $\mathcal{S}\left(2^{-n},\kappa\right)$ for any $n\in\mathbb{N}^{+}$ as follows:

Theorem 2.1 (Multiple Integral Representation)**.** *For any $n\in\mathbb{N}^{+}$, the density function of $\mathcal{S}\left(2^{-n},\kappa\right)$ can be expressed by*

$$f_{\mathcal{S}(2^{-n},\kappa)}\left(x_n\right)=\int_0^\infty\ldots\int_0^\infty\frac{x_{n-1}}{2\sqrt{\pi x_n^3}}e^{-\frac{x_{n-1}^2}{4x_n}}\times\ldots\times\frac{x_1}{2\sqrt{\pi x_2^3}}e^{-\frac{x_1^2}{4x_2}}\frac{\kappa}{2\sqrt{\pi x_1^3}}e^{-\frac{\kappa^2}{4x_1}}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_{n-1}.\tag{2.1}$$

Proof. In fact, for $n = 1$, $\mathcal{S}\left(\frac{1}{2}, \kappa\right)$ is an inverse gamma distribution with the density

$$f_{\mathcal{S}\left(\frac{1}{2},\kappa\right)}(x) = \frac{\kappa}{2\sqrt{\pi x^3}} e^{-\frac{\kappa^2}{4x}}. \tag{2.2}$$

For $n = 2$, the Laplace transform of $S\left(\frac{1}{4}, \kappa\right)$ can be rewritten by

$$\begin{aligned}
\exp\left(-\kappa u^{\frac{1}{4}}\right) &= \exp\left(-\kappa\sqrt{\sqrt{u}}\right) \\
&= \int_0^\infty e^{-\sqrt{u}x_1} \frac{\kappa}{2\sqrt{\pi x_1^3}} \exp\left(-\frac{\kappa^2}{4x_1}\right) \mathrm{d}x_1 \\
&= \int_0^\infty \int_0^\infty e^{-ux_2} \frac{x_1}{2\sqrt{\pi x_2^3}} \exp\left(-\frac{x_1^2}{4x_2}\right) \frac{\kappa}{2\sqrt{\pi x_1^3}} \exp\left(-\frac{\kappa^2}{4x_1}\right) \mathrm{d}x_1 \mathrm{d}x_2,
\end{aligned}$$

which means that the density of $\mathcal{S}\left(\frac{1}{4}, \kappa\right)$ is

$$f_{\mathcal{S}\left(\frac{1}{4},\kappa\right)}(x_2) = \int_0^\infty \frac{x_1}{2\sqrt{\pi x_2^3}} e^{-\frac{x_1^2}{4x_2}} \frac{\kappa}{2\sqrt{\pi x_1^3}} e^{-\frac{\kappa^2}{4x_1}} \mathrm{d}x_1. \tag{2.3}$$

Hence, (2.1) holds for $n = 2$. Let us assume that, this statement holds for any integer smaller than an arbitrary integer $j > 2$, then, for any $n = j + 1$ in general, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-uS\left(2^{-(j+1)},\kappa\right)}\right] &= \exp\left(-\kappa u^{2^{-(j+1)}}\right) \\
&= \exp\left(-\kappa\sqrt{u^j}\right) \\
&= \int_0^\infty e^{-u^{2^{-j}}x_1} \frac{\kappa}{2\sqrt{\pi x_1^3}} e^{-\frac{\kappa^2}{4x_1}} \mathrm{d}x_1 \\
&= \int_0^\infty \int_0^\infty e^{-ux_j} f_{S\left(2^{-(j+1)},x_1\right)}(x_{j+1}) \frac{\kappa}{2\sqrt{\pi x_1^3}} e^{-\frac{\kappa^2}{4x_1}} \mathrm{d}x_1 \mathrm{d}x_{j+1},
\end{aligned}$$

where

$$f_{\mathcal{S}\left(2^{-(j+1)},x_1\right)}(x_{j+1}) = \int_0^\infty \ldots \int_0^\infty \frac{x_j}{2\sqrt{\pi x_{j+1}^2}} e^{-\frac{x_j^2}{4x_{j+1}}} \times \ldots \times \frac{x_1}{2\sqrt{\pi x_2^2}} e^{-\frac{x_1^2}{4x_2}} \mathrm{d}x_2 \ldots \mathrm{d}x_j.$$

Hence, $f_{\mathcal{S}\left(2^{-(j+1)},x_1\right)}(x_1)$ immediately follows (2.1) with $n = j + 1$. Thus, the associated proof can be completed conventionally by the mathematical induction. □

Remark 2.1. Based on Theorem 2.1, we can calculate the density functions of $\mathcal{S}(2^{-n}, \kappa)$ using numerical integration. For example, the density functions of $\mathcal{S}\left(\frac{1}{4}, \kappa\right)$ as specified in (2.3) when $x_2 \in [0, 5]$ for $\kappa = 0.5, 1, 2, 4$ are plotted in Figure 1, respectively.

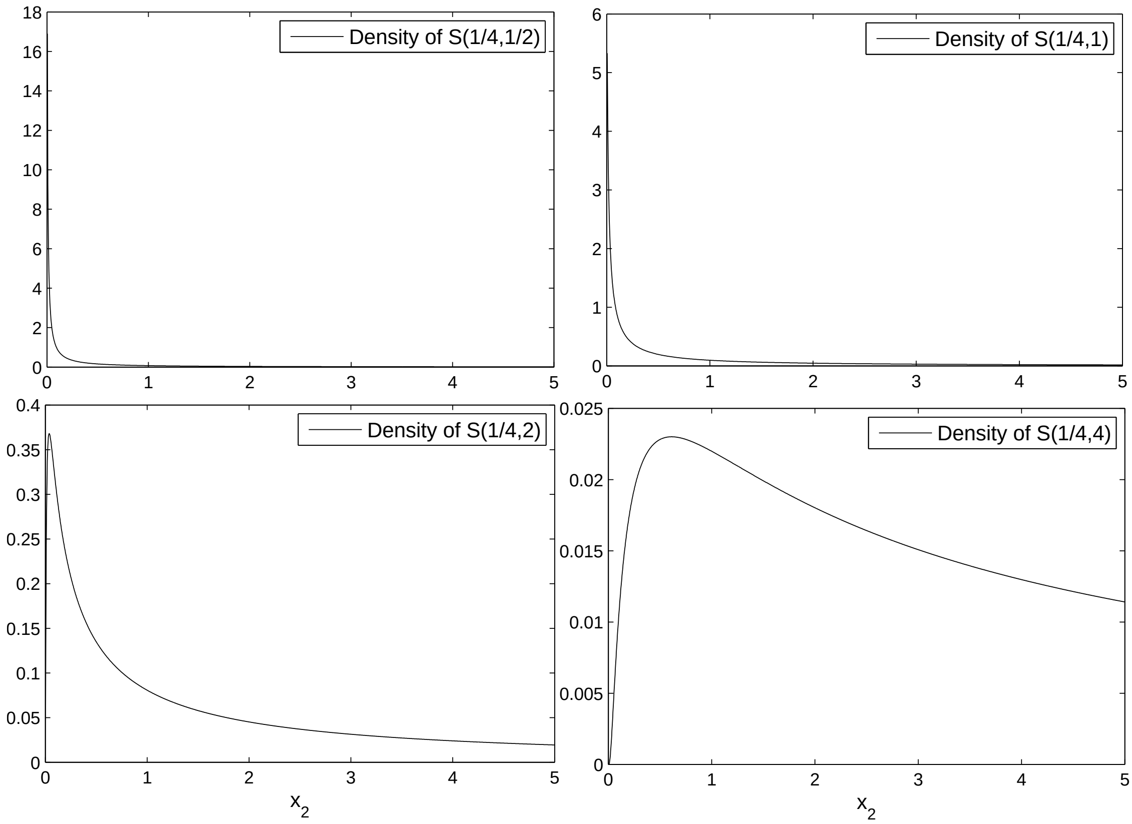

Figure 1: The density functions of $\mathcal{S}\left(\frac{1}{4},\kappa\right)$ for $\kappa = 0.5, 1, 2, 4$, respectively

Moreover, the multiple integral representation for the density function in Theorem 2.1 reveals an elegant distributional decomposition for $\mathcal{S}\left(2^{-n},\kappa\right)$ into simple inverse gamma distributions in a recursive structure as below:

Theorem 2.2 (Distributional Decomposition)**.** *$\mathcal{S}\left(2^{-n},\kappa\right)$ is equal in distribution to a random variable X_n satisfying*

$$X_1 \sim \textsf{InvGamma}\left(\frac{1}{2}, \frac{\kappa^2}{4}\right), \qquad X_i \mid X_{i+1} = x_{i+1} \sim \textsf{InvGamma}\left(\frac{1}{2}, \frac{x_{i+1}^2}{4}\right), \tag{2.4}$$

for $i = 1, ..., n-1$ and $n = 2, 3, ...$.

Proof. Since (2.1) in Theorem 2.1 is the density of X_n, we have $\mathcal{S}\left(2^{-n},\kappa\right) \stackrel{\mathcal{D}}{=} X_n$ for any $n \in \mathbb{N}^+$. $\square$

Remark 2.2. The elegant distributional decomposition of Theorem 2.2 can be numerically verified by Monte Carlo simulation, since itself implies a simple procedure in a backward recursive structure for exactly sampling $\mathcal{S}\left(2^{-n},\kappa\right)$. For example, the comparison between the true values (1.1) and the associated simulation-based estimations (via Theorem 2.2) for the Laplace transform $\mathbb{E}\left[e^{-u\mathcal{S}\left(2^{-n},\kappa\right)}\right]$ with $u = 0.25$ and $\kappa = 0.1, 0.2, ..., 1.0$ based on $100,000$ replications for

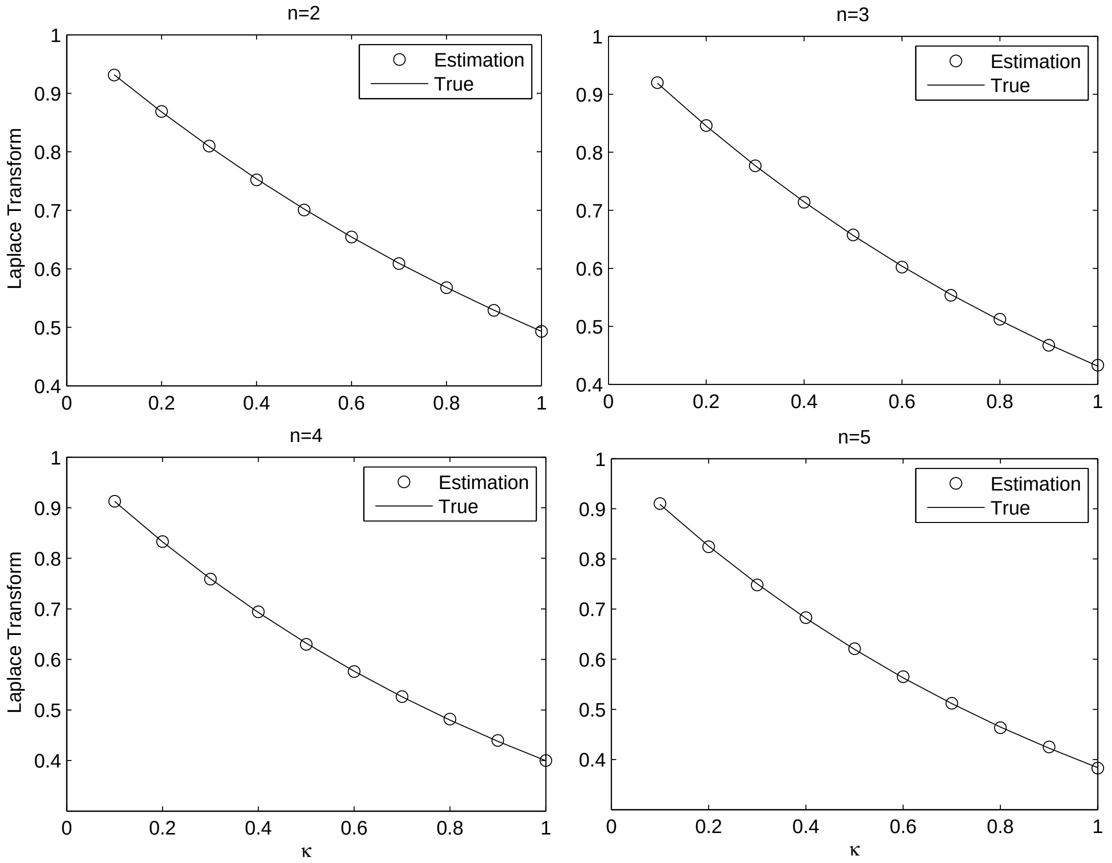

Figure 2: Comparison between the true values (1.1) and the associated simulation-based estimations (via Theorem 2.2) for the Laplace transform $\mathbb{E}\left[e^{-u\mathcal{S}(2^{-n},\kappa)}\right]$ with $u = 0.25$ and $\kappa = 0.1, 0.2, ..., 1.0$ based on $100,000$ replications for $n = 2, 3, 4, 5$, respectively

$n = 2, 3, 4, 5$ is plotted in Figure 2, respectively, and the numerical results are reported in Table 1. They are all conducted on a desktop with an Intel Core i7-6700 CPU@3.40GHz processor, 24.00GB RAM, Windows 10 Professional and 64-bit Operating System. The algorithms are coded and performed in MatLab (R2012a), and the computation time is measured by the *elapsed CPU time* in seconds. The associated errors with respect to the true values calculated by (1.1) are reported by three standard measures:

1. *Error* = estimated value − true value;
2. *Relative error (error %)* $= \dfrac{\text{estimated value} - \text{true value}}{\text{true value}}$;
3. *SE* is the standard error of simulation output.

References

Borak, S., Härdle, W., and Weron, R. (2011). Stable distributions. In Cizek, P., Härdle, W. K., and Weron, R., editors, *Statistical Tools for Finance and Insurance*, pages 21–44. Springer, Berlin.

Table 1: Comparison between the true values (1.1) and the associated simulation-based estimations (via Theorem 2.2) for the Laplace transform $\mathbb{E}\left[e^{-u\mathcal{S}(2^{-n},\kappa)}\right]$ with $u = 0.25$ and $\kappa = 0.1, 0.2, ..., 1.0$ based on $100,000$ replications for $n = 2, 3, 4, 5$, respectively; the associated plots are provided in Figure 2

κ	True	Simulation	Error	Error%	SE	True	Simulation	Error	Error%	SE
			$n=2$					$n=3$		
0.1	0.9317	0.9311	- 0.0006	-0.06%	0.0007	0.9193	0.9200	0.0006	0.07%	0.0008
0.2	0.8681	0.8689	0.0008	0.09%	0.0010	0.8452	0.8461	0.0009	0.11%	0.0011
0.3	0.8089	0.8097	0.0009	0.11%	0.0011	0.7770	0.7767	- 0.0004	-0.05%	0.0012
0.4	0.7536	0.7519	- 0.0018	-0.23%	0.0012	0.7144	0.7139	- 0.0005	-0.07%	0.0014
0.5	0.7022	0.7005	- 0.0017	-0.24%	0.0013	0.6568	0.6575	0.0008	0.12%	0.0014
0.6	0.6543	0.6542	- 0.0000	0.00%	0.0013	0.6038	0.6020	- 0.0018	-0.30%	0.0015
0.7	0.6096	0.6090	- 0.0006	-0.09%	0.0014	0.5551	0.5536	- 0.0015	-0.27%	0.0015
0.8	0.5680	0.5677	- 0.0003	-0.04%	0.0014	0.5103	0.5124	0.0020	0.40%	0.0015
0.9	0.5292	0.5290	- 0.0002	-0.05%	0.0014	0.4692	0.4674	- 0.0017	-0.37%	0.0015
1.0	0.4931	0.4930	- 0.0001	-0.02%	0.0014	0.4313	0.4331	0.0018	0.41%	0.0015
			$n=4$					$n=5$		
0.1	0.9124	0.9128	0.0004	0.05%	0.0009	0.9087	0.9103	0.0017	0.18%	0.0009
0.2	0.8324	0.8330	0.0006	0.07%	0.0012	0.8257	0.8243	- 0.0014	-0.17%	0.0012
0.3	0.7595	0.7588	- 0.0007	-0.09%	0.0013	0.7503	0.7482	- 0.0021	-0.28%	0.0014
0.4	0.6929	0.6942	0.0013	0.18%	0.0014	0.6818	0.6829	0.0011	0.16%	0.0015
0.5	0.6322	0.6298	- 0.0025	-0.39%	0.0015	0.6195	0.6210	0.0014	0.23%	0.0015
0.6	0.5768	0.5760	- 0.0008	-0.15%	0.0015	0.5630	0.5650	0.0020	0.36%	0.0015
0.7	0.5263	0.5263	0.0001	0.01%	0.0015	0.5115	0.5122	0.0007	0.14%	0.0016
0.8	0.4802	0.4819	0.0018	0.37%	0.0015	0.4648	0.4635	- 0.0013	-0.28%	0.0016
0.9	0.4381	0.4399	0.0018	0.40%	0.0015	0.4224	0.4249	0.0025	0.60%	0.0015
1.0	0.3997	0.4000	0.0003	0.07%	0.0015	0.3838	0.3828	- 0.0010	-0.26%	0.0015

Brown, G. W. and Tukey, J. W. (1946). Some distributions of sample means. *Annals of Mathematical Statistics*, 17(1):1–12.

Cizek, P., Härdle, W. K., and Weron, R. (2011). *Statistical Tools for Finance and Insurance*. Springer, Berlin, 2nd edition.

Cont, R., Potters, M., and Bouchaud, J.-P. (1997). Scaling in stock market data: stable laws and beyond. In *Scale Invariance and Beyond*, pages 75–85. Springer.

Fama, E. F. (1965). The behavior of stock-market prices. *Journal of Business*, 38(1):34–105.

Jondeau, E., Poon, S.-H., and Rockinger, M. (2007). *Financial Modeling Under Non-Gaussian Distributions*. Springer.

Lévy, P. (1925). *Calcul des Probabilités*. Gauthier-Villars, Paris.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36(4):394–419.

McCulloch, J. H. (1996). Financial applications of stable distributions. In *Statistical Methods in Finance*, volume 14 of *Handbook of Statistics*, pages 393–425. Elsevier.

Mitra, S. S. (1981). Distribution of symmetric stable laws of index 2^{-n}. *Annals of Probability*, 9(4):710–711.

Mitra, S. S. (1982). Stable laws of index 2^{-n}. *Annals of Probability*, 10(3):857–859.

Mitra, S. S. (1983). Acknowledgment of priority: "Distribution of Symmetric Stable Laws of Index 2^{-n}" and "Stable Laws of Index 2^{-n}". *Annals of Probability*, 11(2):456.

Samoradnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. CRC Press.

Uchaikin, V. V. and Zolotarev, V. M. (1999). *Chance and Stability: Stable Distributions and Their Applications*. VSP.