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A RISK MODEL WITH DELAYED CLAIMS

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Abstract

In this paper we introduce a simple risk model with delayed claims, an extension of the classical Poisson model. The claims are assumed to arrive according to a Poisson process and claims follow a light-tailed distribution, and each loss payment of the claims will be settled with a random period of delay. We obtain asymptotic expressions for the ruin probability by exploiting a connection to Poisson models that are not time homogeneous. A finer asymptotic formula is obtained for the special case of exponentially delayed claims and an exact formula is obtained when the claims are also exponentially distributed.

Keywords: Delayed claim; risk model; ruin probability; asymptotics; generalised Cramér–Lundberg approximation; nonhomogeneous Poisson process

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Secondary 60G55; 60F05

1. Introduction

In a variety of real situations, claims could have already occurred but not been settled or reported immediately. Many factors may lead to the delay of the actual loss payment of the claims. For instance, acronyms, such as IBNR (incurred but not reported) and RBNS (reported but not settled), are typically used to classify the different reasons for delayed claims.

In the literature, the issues around ruin problems involving delayed claim settlements have been studied. Waters and Papatriandafylou (1985) and Trufin *et al.* (2011) considered a discrete-time model for a risk process allowing delayed claims. Boogaert and Haezendonck (1989) discussed a liability process with settling delay in the framework of an economic environment. Yuen *et al.* (2005) introduced a continuous-time model in which one claim is settled immediately and another claim (called a 'by-claim') is settled with delay each time a claim occurs. Delaying claims have also been modelled by a Poisson shot noise process (see Klüppelberg and Mikosch (1995) and Brémaud (2000)), and by a shot noise Cox process (see Macci and Torrisi (2004) and Albrecher and Asmussen (2006)).

In this paper we introduce a simple delayed-claim model. We assume that claims arrive according to a Poisson process, that claims follow a light-tailed distribution, i.e. the distribution of claims has a moment generating function, and that each of the claims will be settled in a randomly delayed period of time. The loss of each claim payment occurs only at the settlement time, rather than at the arrival time. In particular, we consider the special case of exponential delay, where the ultimate ruin probability and asymptotics can be exactly obtained by a power series; this case is a simplified version of the model considered in Yuen *et al.* (2005) without the immediate settled claims.

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The paper is organised as follows. In Section 2 we introduce our model setting for the delayed-claim risk process and the underlying processes of the claim arrival, delay, and settlement. In Section 3 we derive an asymptotic formula for the ruin probability in the general case of delay, and, in particular, exploit a well-known connection to the nonhomogeneous Poisson model. For the special case of exponential delay, the Laplace transform of the nonruin probability and a finer asymptotic expansion for the ruin probability are obtained in Section 4. In Section 5 we derive an exact formula for the ruin probability by assuming that the claims are exponentially delayed and the sizes are exponentially distributed.

2. Risk process

Consider a surplus process $\{X_t\}_{t\geq 0}$ in continuous time on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e.

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \qquad t \geq 0,$$

where

- $x = X_0 \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is the number of cumulative settled claims within the time interval $[0, t]$ (assume that $N_0 = 0$);
- $\{Z_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed positive random variables (claim sizes), independent of N_t, following a light-tailed distribution with cumulative distribution function $Z(z)$, $z > 0$, i.e.

$$\hat{z}(w) = \int_0^\infty \mathrm{e}^{-wz}\,\mathrm{d}Z(z) < \infty \quad \text{for some } w < 0;$$

  the mean and tail of Z are respectively denoted by

$$\mu_{1_Z} = \int_0^\infty z\,\mathrm{d}Z(z), \qquad \bar{Z}(x) = \int_x^\infty \mathrm{d}Z(s).$$

Assume that the arrival of claims follows a Poisson process of rate ρ, and that each of the claims will be settled with a random delay. Loss occurs only when claims are being settled. Denote by M_t the number of cumulative unsettled claims within the time interval $[0, t]$ and assume that the initial number $M_0 = 0$. Let $\{T_k\}_{k=1,2,\ldots}$, $\{L_k\}_{k=1,2,\ldots}$, and $\{T_k + L_k\}_{k=1,2,\ldots}$ denote the (random) times of the claim arrival, delay, and settlement, respectively, and, hence,

$$M_t = \sum_k (\mathbf{1}\{T_k \leq t\} - \mathbf{1}\{T_k + L_k \leq t\}), \qquad N_t = \sum_k \mathbf{1}\{T_k + L_k \leq t\};$$

$\{L_k\}_{k=1,2,\ldots}$ are independent and identically distributed nonnegative random variables with cumulative distribution function L. A sample path of the joint point processes of the cumulative settled and unsettled claims (N_t, M_t) is given by Figure 1.

The ruin (stopping) time after time $t \geq 0$ is defined by

$$\tau_t^* := \begin{cases} \inf\{s : s > t,\ X_s \leq 0\} & \\ \inf\{\varnothing\} = \infty & \text{if } X_s > 0 \text{ for all } s; \end{cases}$$

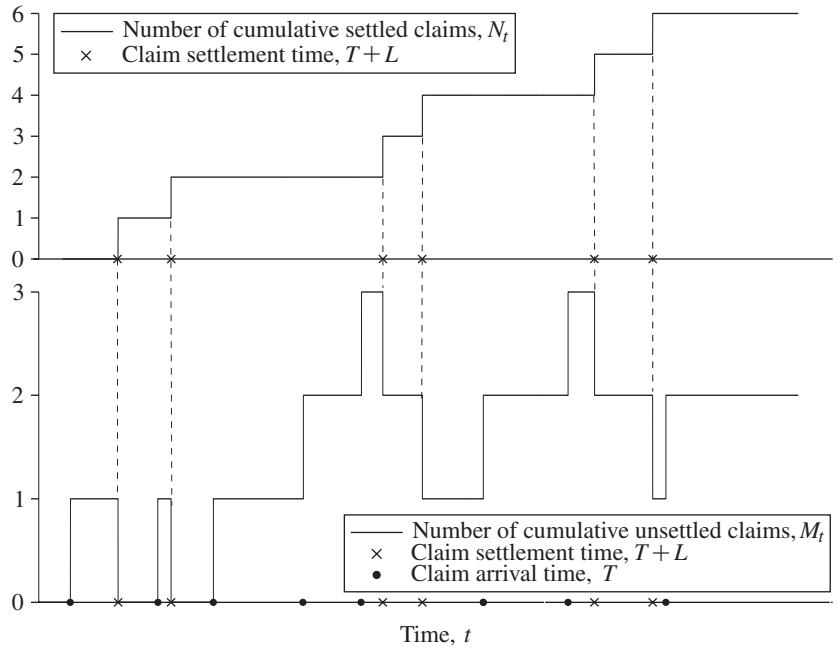

FIGURE 1: A sample path of the joint point processes of cumulative settled and unsettled claims (N_t, M_t).

in particular, $\tau_t^* = \infty$ means that ruin does not occur. We are interested in the ultimate ruin probability at time t, i.e.

$$\psi(x,t) =: \mathbb{P}\{\tau_t^* < \infty \mid X_t = x\}, \tag{1}$$

or, the ultimate nonruin probability at time t, i.e.

$$\phi(x,t) =: 1 - \psi(x,t). \tag{2}$$

Note that $\psi(x,t)$ defined in (1) is the ultimate ruin probability at the general time $t \geq 0$, rather than the conventionally defined ruin probability of the finite-horizon time t.

3. Ruin with randomly delayed claims

3.1. Preliminaries

The net profit condition remains the same as in the classical Poisson model, i.e. $c > \rho\mu_{1_Z}$, since, obviously, $\lim_{t\to\infty} \int_0^t \bar{L}(s)\,\mathrm{d}s/t = 0$, and

$$\begin{aligned}
\lim_{t\to\infty} \frac{\mathbb{E}[X_t]}{t} &= \lim_{t\to\infty} \frac{x + ct - \mu_{1_Z}\mathbb{E}[N_t]}{t} \\
&= \lim_{t\to\infty} \frac{x + ct - \mu_{1_Z}\rho(t - \int_0^t \bar{L}(s)\,\mathrm{d}s)}{t} \\
&= c - \rho\mu_{1_Z} \\
&> 0.
\end{aligned}$$

Lemma 1. *Assume that $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$. Then we have a series of modified Lundberg fundamental equations:*

$$cw - \rho[1 - \hat{z}(w)] - \delta j = 0, \qquad j = 0, 1, \ldots. \tag{3}$$

- *For $j = 0$, (3) has solution 0 and a unique negative solution (denoted by $W_0^+ = 0$ and $W_0^- < 0$).*
- *For $j = 1, 2, \ldots$, (3) has unique positive and negative solutions (denoted by $W_j^+ > 0$ and $W_j^- < 0$).*

Proof. Rewrite (3) as

$$\hat{z}(w) = l_j(w), \tag{4}$$

where $l_j(w) =: -cw/\rho + (1+\delta j/\rho)$, $j = 0, 1, \ldots$. Note that $Z(z)$ is a light-tailed distribution, and

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0} = -\mu_{1_Z}, \qquad \left.\frac{\mathrm{d}l_j(w)}{\mathrm{d}w}\right|_{w=0} = -\frac{c}{\rho};$$

by the net profit condition $c > \rho\mu_{1_Z}$, we have

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0} > \left.\frac{\mathrm{d}l_j(w)}{\partial w}\right|_{w=0}.$$

In particular, for $j = 0$, we have $l_0(0) = \hat{z}(0) = 1$. Then, further by the convexity of $\hat{z}(w)$ and the linearity of $l_j(w)$, the uniqueness of the positive and negative solutions to (3) follows immediately. As an illustration, a plot of (4) is given in Figure 2.

Denote the (modified) adjustment coefficients by $R_j =: -W_j^-$, $j = 0, 1, \ldots$; note that $0 < R_0 < R_1 < R_2 < \cdots < R_\infty$, where $R_\infty =: \inf\{R \mid \hat{z}(-R) = \infty\}$.

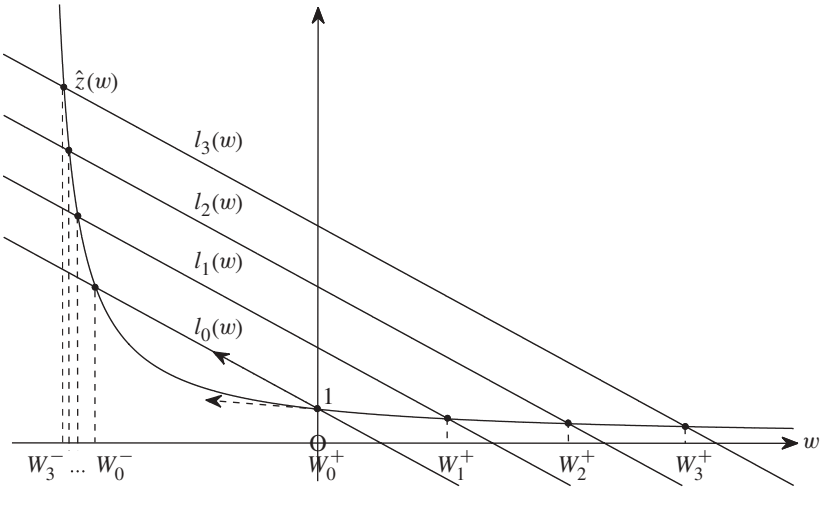

FIGURE 2: Lundberg's fundamental equations.

Example 1. If $Z \sim \mathrm{Exp}(\gamma)$ then we have a series of modified Lundberg fundamental equations $cw^2 + (c\gamma - \rho - \delta j)w - \gamma\delta j = 0,\ j = 0, 1, \ldots$, with explicit solutions

$$W_j^{\pm} = \frac{\rho + \delta j - c\gamma \pm \sqrt{(\rho + \delta j - c\gamma)^2 + 4c\gamma\delta j}}{2c}, \qquad j = 0, 1, \ldots,$$

and $R_\infty = \lim_{j\to\infty} R_j = \gamma$.

3.2. Asymptotics of the ruin probability

From Mirasol (1963) we know that a delayed (or displaced) Poisson process is still a (nonhomogeneous) Poisson process, which is also a special case of the discretised dynamic contagion process introduced in Dassios and Zhao (2012); see also Newell (1966), Lawrance and Lewis (1975), and Dassios and Zhao (2011). According to the model setting in Section 2, the settlement process N_t is a nonhomogeneous Poisson process with rate $\rho L(t)$, and we can obtain the asymptotics of the ruin probability as follows.

Theorem 1. *Assume that $c > \rho\mu_{1Z}$, and that the first and second moments of L exist. Then the asymptotics of ruin probability are given by*

$$\psi(x,t) \sim \exp\left(-cR_0 \int_t^\infty \bar{L}(s)\,\mathrm{d}s\right) \frac{c - \rho\mu_{1Z}}{\rho \int_0^\infty z\mathrm{e}^{R_0 z}\,\mathrm{d}Z(z) - c} \mathrm{e}^{-R_0 x} + o(\mathrm{e}^{-R_0 x}) \quad \textit{as } x \to \infty,$$

where $\bar{L}(t) =: 1 - L(t)$.

Proof. The ruin probability $\psi(x,t)$ defined in (1) is the probability of ultimate ruin when the current reserve is x at current time t. In a sufficiently small time period Δt after time t, $[t, t + \Delta t]$, we observe the following.

(i) No claim occurs with probability $1 - \rho L(t)\Delta t$, ruin does not occur, and $X_{t+\Delta t} = x + c\Delta t$.

(ii) One claim Z occurs with probability $\rho L(t)\Delta t$. Then $X_{t+\Delta t} = x + c\Delta t - Z$ and we have two possibilities:

(ii.1) ruin has not occurred if $Z \le x + c\Delta t$;

(ii.2) ruin has occurred if $Z > x + c\Delta t$.

Using the Markov property, we have

$$\begin{aligned}\psi(x,t) &= (1 - \rho L(t)\Delta t)\psi(x + c\Delta t, t + \Delta t)\\ &\quad + \rho L(t)\Delta t\left(\int_0^{x+c\Delta t} \psi(x + c\Delta t - z, t + \Delta t)\,\mathrm{d}Z(z) + [1 - Z(x + c\Delta t)]\right)\\ &\quad + o(\Delta t).\end{aligned}$$

Since

$$\psi(x + c\Delta t, t + \Delta t) = \psi(x,t) + c\Delta t\frac{\partial\psi(x,t)}{\partial x} + \frac{\partial\psi(x,t)}{\partial t}\Delta t + o(\Delta t),$$

the integrodifferential equation of $\psi(x,t)$ is then given by

$$\frac{\partial\psi(x,t)}{\partial t} + c\frac{\partial\psi(x,t)}{\partial x} + \rho L(t)\left(\int_0^x \psi(x - z, t)\,\mathrm{d}Z(z) + \bar{Z}(x) - \psi(x,t)\right) = 0.$$

By the Laplace transform $\hat{\psi}(w,t) =: \mathcal{L}_w\{\psi(x,t)\} = \int_0^\infty \mathrm{e}^{-wx}\psi(x,t)\,\mathrm{d}x$, we have

$$\frac{\partial\hat{\psi}(w,t)}{\partial t} - c\psi(0,t) + (cw - \rho L(t)[1-\hat{z}(w)])\hat{\psi}(w,t) + \rho L(t)\frac{1-\hat{z}(w)}{w} = 0. \tag{5}$$

Define

$$\hat{\psi}(w,t) =: \frac{\rho(\mu_{1_Z} - (1-\hat{z}(w))/w)}{cw - \rho[1-\hat{z}(w)]}\exp\biggl(\rho\int_t^\infty [1-\hat{z}(w)]\bar{L}(s)\,\mathrm{d}s\biggr) + \hat{k}(w,t), \tag{6}$$

where $\hat{k}(w,t)$ is the Laplace transform of a function $k(x,t)$ and satisfies

$$\lim_{t\to\infty}\hat{k}(w,t) = 0. \tag{7}$$

Substituting (6) into (5) yields the ordinary differential equation of $\hat{k}(w,t)$:

$$\begin{aligned}
&\frac{\partial\hat{k}(w,t)}{\partial t} + (cw - \rho[1-\hat{z}(w)] + \rho\bar{L}(t)[1-\hat{z}(w)])\hat{k}(w,t)\\
&\quad = c\biggl(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\biggr) + \rho\biggl(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\biggr)\biggl(\exp\biggl(\rho\int_t^\infty [1-\hat{z}(w)]\bar{L}(s)\,\mathrm{d}s\biggr) - 1\biggr)\\
&\qquad + \rho\bar{L}(t)\frac{1-\hat{z}(w)}{w}.
\end{aligned}$$

By multiplying $\mathrm{e}^{(cw-\rho[1-\hat{z}(w)])t}\exp(-\rho\int_t^\infty \bar{L}(s)[1-\hat{z}(w)]\,\mathrm{d}s)$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\biggl(\hat{k}(w,t)\mathrm{e}^{(cw-\rho[1-\hat{z}(w)])t}\exp\biggl(-\rho\int_t^\infty \bar{L}(s)[1-\hat{z}(w)]\,\mathrm{d}s\biggr)\biggr)\\
&\quad = \biggl[c\biggl(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\biggr) + \rho\biggl(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\biggr)\biggl(\exp\biggl(\rho\int_t^\infty [1-\hat{z}(w)]\bar{L}(s)\,\mathrm{d}s\biggr) - 1\biggr)\\
&\qquad + \rho\bar{L}(t)\frac{1-\hat{z}(w)}{w}\biggr]\mathrm{e}^{(cw-\rho[1-\hat{z}(w)])t}\exp\biggl(-\rho\int_t^\infty \bar{L}(s)[1-\hat{z}(w)]\,\mathrm{d}s\biggr),
\end{aligned}$$

with boundary condition (7). The solution is then

$$\begin{aligned}
\hat{k}(w,t) &= \mathrm{e}^{-(cw-\rho[1-\hat{z}(w)])t}\exp\biggl(\rho\int_t^\infty \bar{L}(s)[1-\hat{z}(w)]\,\mathrm{d}s\biggr)\\
&\quad\times\int_t^\infty \mathrm{e}^{(cw-\rho[1-\hat{z}(w)])s}\exp\biggl(-\rho\int_s^\infty \bar{L}(u)[1-\hat{z}(w)]\,\mathrm{d}u\biggr)\\
&\qquad\times\biggl[-c\biggl(\psi(0,s) - \frac{\rho\mu_{1_Z}}{c}\biggr)\\
&\qquad\quad - \rho\biggl(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\biggr)\biggl(\exp\biggl(\rho\int_s^\infty [1-\hat{z}(w)]\bar{L}(u)\,\mathrm{d}u\biggr) - 1\biggr)\\
&\qquad\quad - \rho\bar{L}(s)\frac{1-\hat{z}(w)}{w}\biggr]\,\mathrm{d}s.
\end{aligned} \tag{8}$$

Obviously, from Figure 2, for $-R_0 < w < 0$, we have $l_0(w) > \hat{z}(w)$, i.e. $cw - \rho[1-\hat{z}(w)] < 0$, $-R_0 < w < 0$. We now respectively discuss the three terms of $\hat{k}(w,t)$ given in (8).

(i) It is well known that (see Gerber (1979) and Grandel (1991)) in the classical model when the claim settlement follows a Poisson process with a constant rate λ, the ruin probability

with initial reserve $x = 0$ is simply $\mu_{1_Z}\lambda/c$, whereas $\psi(0, t)$ here in the first term of (8) is based on the realisation of the rate $\{\rho L(s)\}_{t\le s\le\infty}$. Also, the cumulative function $L(s)$ is an increasing function of s, so the ruin probability $\psi(0, t)$ should be greater than the $\lambda = \rho L(t)$ case and smaller than the $\lambda = \rho L(\infty) = \rho$ case of the classical model, i.e. $\mu_{1_Z}\rho L(t)/c < \psi(0, t) < \mu_{1_Z}\rho/c$, or $0 < \rho\mu_{1_Z}/c - \psi(0, t) < \rho\mu_{1_Z}\bar{L}(t)/c$. If the first moment of L exists then we have

$$\int_t^\infty \left|\psi(0, s) - \frac{\rho\mu_{1_Z}}{c}\right| \mathrm{d}s < \frac{\rho\mu_{1_Z}}{c}\int_t^\infty \bar{L}(s)\,\mathrm{d}s < \frac{\rho\mu_{1_Z}}{c}\int_0^\infty \bar{L}(s)\,\mathrm{d}s < \infty.$$

(ii) For the second term of (8), if the second moment of L exists then

$$\begin{aligned}
&\int_t^\infty \exp\left(-\rho\int_s^\infty [1-\hat{z}(w)]\bar{L}(u)\,\mathrm{d}u\right)\left(\exp\left(\rho\int_s^\infty [1-\hat{z}(w)]\bar{L}(u)\,\mathrm{d}u - 1\right)\mathrm{d}s\right)\\
&\quad = \int_t^\infty \left(1 - \exp\left(-\rho\int_s^\infty [1-\hat{z}(w)]\bar{L}(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&\quad < \int_t^\infty \rho\int_s^\infty [1-\hat{z}(w)]\bar{L}(u)\,\mathrm{d}u\,\mathrm{d}s\\
&\quad < \rho[1-\hat{z}(w)]\int_0^\infty\int_s^\infty \bar{L}(u)\,\mathrm{d}u\,\mathrm{d}s\\
&\quad < \infty.
\end{aligned}$$

(iii) For the third term of (8), if the first moment of L exists then

$$\int_t^\infty \rho\bar{L}(s)\frac{1-\hat{z}(w)}{w}\,\mathrm{d}s = \rho\frac{1-\hat{z}(w)}{w}\int_t^\infty \bar{L}(s)\,\mathrm{d}s < \rho\frac{1-\hat{z}(w)}{w}\int_0^\infty \bar{L}(s)\,\mathrm{d}s < \infty.$$

Therefore, for $-R_0 < w < 0$, we have $\hat{k}(w, t) < \infty$ and

$$\hat{k}(-R_0, t) = \lim_{w\downarrow -R_0}\hat{k}(w, t) = \int_0^\infty \mathrm{e}^{R_0 x}k(x, t)\,\mathrm{d}x < \infty;$$

hence, $k(x, t) = o(\mathrm{e}^{-R_0 x})$. By the final value theorem and $\hat{\psi}(w, t)$ given in (6), we have

$$\begin{aligned}
&\lim_{x\to\infty}\mathrm{e}^{R_0 x}\psi(x, t)\\
&\quad = \lim_{w\to 0} w\mathcal{L}_w\{\mathrm{e}^{R_0 x}\psi(x, t)\}\\
&\quad = \lim_{w\to 0} w\hat{\psi}(w - R_0, t)\\
&\quad = \exp\left(\rho\int_t^\infty [1-\hat{z}(-R_0)]\bar{L}(s)\,\mathrm{d}s\right)\lim_{w\to 0} w\frac{\rho(\mu_{1_Z} - (1-\hat{z}(w-R_0))/(w-R_0))}{c(w-R_0) - \rho[1-\hat{z}(w-R_0)]}\\
&\qquad + \lim_{w\to 0} w\hat{k}(w - R_0, t)\\
&\quad = \exp\left(-cR_0\int_t^\infty \bar{L}(s)\,\mathrm{d}s\right)\frac{c - \rho\mu_{1_Z}}{\rho\int_0^\infty z\mathrm{e}^{R_0 z}\,\mathrm{d}Z(z) - c}.
\end{aligned}$$

Note that, by definition, $-R_0$ is the solution to $cw - \rho[1-\hat{z}(w)] = 0$, and we have $1-\hat{z}(-R_0) = -cR_0/\rho$.

4. Ruin with exponentially delayed claims

By specifying the distribution of the period of delay, L, we could improve the result in Theorem 1 with higher-order asymptotics. Here, for instance, we consider the special case when the claims are exponentially delayed, say $L \sim \mathrm{Exp}(\delta)$, in order to derive $o(\mathrm{e}^{-R_0 x})$ in more detail.

4.1. Laplace transform of the nonruin probability

We derive the Laplace transform of the nonruin probability in two different expressions, given in Theorem 2 and Theorem 3, respectively, which we then use to derive the asymptotics of the ruin probability.

Theorem 2. *Assume that $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$. Then the Laplace transform of the nonruin probability is given by*

$$\hat{\phi}(w,t) = \mathrm{e}^{\vartheta \mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\left(\frac{c-\rho\mu_{1_Z}}{cw-\rho[1-\hat{z}(w)]} + c\sum_{j=1}^{\infty}\mathrm{e}^{-j\delta t}\frac{\sum_{\ell=0}^{j} r_\ell[\vartheta\hat{z}(w)]^{j-\ell}/(j-\ell)!}{cw-\rho[1-\hat{z}(w)]-\delta j}\right), \tag{9}$$

where $\vartheta = \rho/\delta$,

$$r_0 = 1-\frac{\rho}{c}\mu_{1_Z}, \qquad r_\ell = -\sum_{i=0}^{\ell-1}\frac{[\vartheta\hat{z}(W_\ell^+)]^{\ell-i}}{(\ell-i)!}r_i, \qquad \ell = 1,2,\ldots. \tag{10}$$

Proof. If $L \sim \mathrm{Exp}(\delta)$ then $L(t) = 1-\mathrm{e}^{-\delta t}$, N_t is a nonhomogeneous Poisson process with rate $\rho - \vartheta\delta\mathrm{e}^{-\delta t}$, and the nonruin probability $\phi(x,t)$ defined in (2) satisfies the integrodifferential equation

$$\frac{\partial\phi(x,t)}{\partial t} + c\frac{\partial\phi(x,t)}{\partial x} + (\rho-\vartheta\delta\mathrm{e}^{-\delta t})\left(\int_0^x \phi(x-z,t)\,\mathrm{d}Z(z) - \phi(x,t)\right) = 0.$$

By the Laplace transform

$$\hat{\phi}(w,t) =: \mathcal{L}_w\{\phi(x,t)\} = \int_0^\infty \mathrm{e}^{-wx}\phi(x,t)\,\mathrm{d}x,$$

we have

$$\frac{\partial\hat{\phi}(w,t)}{\partial t} + c(w\hat{\phi}(w,t)-\phi(0,t)) - (\rho-\vartheta\delta\mathrm{e}^{-\delta t})[1-\hat{z}(w)]\hat{\phi}(w,t) = 0. \tag{11}$$

Define

$$\hat{h}(w,t) =: \hat{\phi}(w,t)\exp\left(\int_0^t \delta\vartheta\mathrm{e}^{-\delta s}[1-\hat{z}(w)]\,\mathrm{d}s\right),$$

where $\hat{h}(w,t)$ is the Laplace transform of a function $h(x,t)$. Then

$$\hat{\phi}(w,t) = \hat{h}(w,t)\mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})[1-\hat{z}(w)]}. \tag{12}$$

Substituting (12) into (11) yields

$$\frac{\partial\hat{h}(w,t)}{\partial t} + c(w\hat{h}(w,t)-\phi(0,t)\mathrm{e}^{\vartheta(1-\mathrm{e}^{-\delta t})[1-\hat{z}(w)]}) - \rho[1-\hat{z}(w)]\hat{h}(w,t) = 0. \tag{13}$$

Note that by (12) we have

$$\begin{aligned}\hat{\phi}(w,t) &= \hat{h}(w,t)\mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})}\mathrm{e}^{\vartheta(1-\mathrm{e}^{-\delta t})\hat{z}(w)}\\ &= \mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})}\left(\hat{h}(w,t)+\sum_{k=1}^{\infty}\frac{(\vartheta(1-\mathrm{e}^{-\delta t}))^k}{k!}\hat{h}(w,t)\hat{z}^k(w)\right),\end{aligned}$$

which is the Laplace transform of

$$\phi(x,t) = \mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})}\left(h(x,t)+\sum_{k=1}^{\infty}\frac{(\vartheta(1-\mathrm{e}^{-\delta t}))^k}{k!}\int_0^x h(x-z,t)\,\mathrm{d}Z^{(k)}(z)\right),$$

where $Z^{(k)}$ is the k-fold convolution of the distribution Z, i.e. $Z^{(k)} \stackrel{\mathrm{D}}{=} \sum_{i=1}^{k} Z_i$. Then we have

$$\phi(0,t) = h(0,t)\mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})}. \tag{14}$$

Substituting (14) into (13) yields

$$\frac{\partial\hat{h}(w,t)}{\partial t} + (cw-\rho[1-\hat{z}(w)])\hat{h}(w,t) - c\mathrm{e}^{-\vartheta\hat{z}(w)}h(0,t)\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}\hat{z}(w)} = 0.$$

This equation of $\hat{h}(w,t)$ has a power series solution $\hat{h}(w,t) = \sum_{j=0}^{\infty}\mathrm{e}^{-j\delta t}\hat{h}_j(w)$, the Laplace transform of $h(x,t) = \sum_{j=0}^{\infty}\mathrm{e}^{-j\delta t}h_j(x)$. Since

$$\begin{aligned}\frac{\partial\hat{h}(w,t)}{\partial t} &= -\delta\sum_{j=0}^{\infty} j\mathrm{e}^{-j\delta t}\hat{h}_j(w),\\ h(0,t)\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}\hat{z}(w)} &= \sum_{j=0}^{\infty}\mathrm{e}^{-j\delta t}h_j(0)\sum_{k=0}^{\infty}\frac{\mathrm{e}^{-k\delta t}[\vartheta\hat{z}(w)]^k}{k!}\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{e}^{-(j+k)\delta t}h_j(0)\frac{[\vartheta\hat{z}(w)]^k}{k!} \quad (j+k=i)\\ &= \sum_{i=0}^{\infty}\mathrm{e}^{-i\delta t}\sum_{j=0}^{i}h_j(0)\frac{[\vartheta\hat{z}(w)]^{i-j}}{(i-j)!},\end{aligned}$$

we have

$$\sum_{j=0}^{\infty}\mathrm{e}^{-j\delta t}\left[(-\delta j+cw-\rho[1-\hat{z}(w)])\hat{h}_j(w) - c\mathrm{e}^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j}h_\ell(0)\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}\right] = 0.$$

Then, for any $j = 0, 1, \ldots,$

$$(-\delta j+cw-\rho[1-\hat{z}(w)])\hat{h}_j(w) - c\mathrm{e}^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j}h_\ell(0)\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!} = 0,$$

and, hence,

$$\hat{h}_j(w) = \frac{c\mathrm{e}^{-\vartheta\hat{z}(w)}}{cw-\rho[1-\hat{z}(w)]-\delta j}\sum_{\ell=0}^{j}h_\ell(0)\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \qquad j = 0, 1, \ldots. \tag{15}$$

Note that the denominator in (15) is the modified Lundberg fundamental equation given in Lemma 1.

By (12) we have

$$\hat{\phi}(w,t) = \mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})[1-\hat{z}(w)]}\left(\hat{h}_0(w) + \sum_{j=1}^{\infty} \mathrm{e}^{-j\delta t}\hat{h}_j(w)\right). \tag{16}$$

Note that if $t \to \infty$, we recover the classical Poisson model. By (16) we have

$$\hat{\phi}(w,\infty) = \mathrm{e}^{-\vartheta[1-\hat{z}(w)]}\hat{h}_0(w), \tag{17}$$

$$\hat{\phi}(w,0) = \sum_{j=0}^{\infty} \hat{h}_j(w).$$

The series of constants $\{h_\ell(0)\}_{\ell=0,1,\ldots}$ in (15) can be obtained as follows.

For $j = 0$, by (15) we have

$$\hat{h}_0(w) = \frac{c\mathrm{e}^{-\vartheta\hat{z}(w)}}{cw - \rho[1-\hat{z}(w)]}h_0(0).$$

By (14) and (17), we have

$$\phi(0,\infty) = h(0,\infty)\mathrm{e}^{-\vartheta} = h_0(0)\mathrm{e}^{-\vartheta},$$

$$\hat{\phi}(w,\infty) = \hat{h}_0(w)\mathrm{e}^{-\vartheta[1-\hat{z}(w)]} = \frac{c\mathrm{e}^{-\vartheta}h_0(0)}{cw - \rho[1-\hat{z}(w)]} = \frac{c\phi(0,\infty)}{cw - \rho[1-\hat{z}(w)]}.$$

Since

$$\lim_{x\to\infty}\phi(x,\infty) = \lim_{w\to 0} w\hat{\phi}(w,\infty) = 1,$$

i.e.

$$\lim_{w\to 0} w\frac{c\phi(0,\infty)}{cw - \rho[1-\hat{z}(w)]} = \frac{c\phi(0,\infty)}{\lim_{w\to 0} w^{-1}(cw - \rho[1-\hat{z}(w)])} = \frac{c\phi(0,\infty)}{c - \rho\mu_{1_Z}} = 1,$$

we have $\phi(0,t) = (c - \rho\mu_{1_Z})/c$,

$$h_0(0) = \frac{\mathrm{e}^{\vartheta}(c - \rho\mu_{1_Z})}{c}, \tag{18}$$

and

$$\hat{\phi}(w,t) = \frac{c - \rho\mu_{1_Z}}{cw - \rho[1-\hat{z}(w)]},$$

which is precisely the Laplace transform of the ultimate nonruin probability of the classical Poisson model. Hence, we have

$$\hat{h}_0(w) = \mathrm{e}^{\vartheta[1-\hat{z}(w)]}\frac{c - \rho\mu_{1_Z}}{cw - \rho[1-\hat{z}(w)]}.$$

For $j = 1, 2, \ldots$, since $\hat{h}_j(w)$ of (15) exists at $w = W_j^+$, we have

$$\lim_{w\to W_j^+}\left(c\mathrm{e}^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j} h_\ell(0)\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}\right) = 0, \qquad j = 1, 2, \ldots,$$

or

$$\sum_{\ell=0}^{j}\frac{[\vartheta\hat{z}(W_j^+)]^{j-\ell}}{(j-\ell)!}h_\ell(0) = 0, \qquad j = 1, 2, \ldots.$$

Given the initial value $h_0(0)$ in (18), obviously the series of constants $\{h_\ell(0)\}_{\ell=1,2,\dots}$ can be solved uniquely and explicitly by recursion. Define the solution by $r_j =: \mathrm{e}^{-\vartheta} h_j(0)$, with the initial value $r_0 = 1 - \rho\mu_{1_Z}/c$. We have

$$\hat{h}_j(w) = \frac{c\mathrm{e}^{\vartheta[1-\hat{z}(w)]}}{cw - \rho[1-\hat{z}(w)] - \delta j} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \qquad j = 1, 2, \dots,$$

where

$$r_\ell = -\sum_{i=0}^{\ell-1} \frac{[\vartheta\hat{z}(W_\ell^+)]^{\ell-i}}{(\ell-i)!} r_i, \qquad \ell = 1, 2, \dots.$$

Therefore, by (16) we have the Laplace transform of the nonruin probability

$$\hat{\phi}(w,t) = \mathrm{e}^{-\vartheta(1-\mathrm{e}^{-\delta t})[1-\hat{z}(w)]} \times \left(\frac{\mathrm{e}^{\vartheta[1-\hat{z}(w)]}(c-\rho\mu_{1_Z})}{cw - \rho[1-\hat{z}(w)]} + \sum_{j=1}^{\infty} \mathrm{e}^{-j\delta t} \frac{c\mathrm{e}^{\vartheta[1-\hat{z}(w)]} \sum_{\ell=0}^{j} r_\ell [\vartheta\hat{z}(w)]^{j-\ell}/(j-\ell)!}{cw - \rho[1-\hat{z}(w)] - \delta j} \right).$$

Remark 1. For $t = 0$, we have

$$\hat{\phi}(w,0) = \mathrm{e}^{\vartheta[1-\hat{z}(w)]} \left(\frac{c - \rho\mu_{1_Z}}{cw - \rho[1-\hat{z}(w)]} + c \sum_{j=1}^{\infty} \frac{\sum_{\ell=0}^{j} r_\ell [\vartheta\hat{z}(w)]^{j-\ell}/(j-\ell)!}{cw - \rho[1-\hat{z}(w)] - \delta j} \right)$$

and, for $t = \infty$,

$$\hat{\phi}(w,\infty) = \frac{c - \rho\mu_{1_Z}}{cw - \rho[1-\hat{z}(w)]},$$

which recovers the result of the classical Poisson model.

Remark 2. Equation (10) offers a numerically tractable formula for calculating the $\{r_j\}_{j=0,1,\dots}$ coefficients; e.g. if $Z \sim \mathrm{Exp}(\gamma)$ with parameter setting $(c, \delta, \rho, \gamma) = (1.5, 2.0, 0.5, 1.0)$, then we have $r_0 = 0.6667$, $r_1 = -0.0657$, $r_2 = 0.0028$, and $r_3 = -7.2560 \times 10^{-5}, \dots$.

Alternatively, the Laplace transform of the nonruin probability can be expressed by the following power series.

Theorem 3. *Assume that $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$. Then the Laplace transform of the nonruin probability is given by $\hat{\phi}(w,t) = \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t}\hat{\phi}_j(w)$, where the $\{\hat{\phi}_j(w)\}_{j=0,1,\dots}$ follow the recurrence*

$$\hat{\phi}_j(w) = \rho \frac{[1-\hat{z}(W_j^+)]\hat{\phi}_{j-1}(W_j^+) - [1-\hat{z}(w)]\hat{\phi}_{j-1}(w)}{cw - \rho[1-\hat{z}(w)] - \delta j}, \qquad j = 1, 2, \dots, \tag{19}$$

$$\hat{\phi}_0(w) = \frac{c(1-\rho\mu_{1_Z}/c)}{cw - \rho[1-\hat{z}(w)]}. \tag{20}$$

Proof. Rewrite (11) as

$$\frac{\partial\hat{\phi}(w,t)}{\partial t} + c(w\hat{\phi}(w,t) - \phi(0,t)) - \rho[1-\hat{z}(w)]\hat{\phi}(w,t) + \rho[1-\hat{z}(w)]\mathrm{e}^{-\delta t}\hat{\phi}(w,t) = 0.$$

This equation has a power series solution $\hat{\phi}(w,t) = \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t}\hat{\phi}_j(w)$, the Laplace transform of the nonruin probability $\phi(x,t) = \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t}\phi_j(x)$. Note that by setting $\hat{\phi}_{-1}(w) = 0$

we have

$$\frac{\partial \hat{\phi}(w,t)}{\partial t} = -\delta \sum_{j=0}^{\infty} j \mathrm{e}^{-j\delta t} \hat{\phi}_j(w),$$

$$\mathrm{e}^{-\delta t} \hat{\phi}(w,t) = \sum_{j=0}^{\infty} \mathrm{e}^{-(j+1)\delta t} \hat{\phi}_j(w) = \sum_{j=1}^{\infty} \mathrm{e}^{-j\delta t} \hat{\phi}_{j-1}(w) = \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} \hat{\phi}_{j-1}(w).$$

Then

$$\begin{aligned} &-\delta \sum_{j=0}^{\infty} j \mathrm{e}^{-j\delta t} \hat{\phi}_j(w) + c\left(w \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} \hat{\phi}_j(w) - \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} \phi_j(0) \right) \\ &- \rho[1-\hat{z}(w)] \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} \hat{\phi}_j(w) + \rho[1-\hat{z}(w)] \sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} \hat{\phi}_{j-1}(w) \\ &\quad = 0, \end{aligned}$$

or

$$\sum_{j=0}^{\infty} \mathrm{e}^{-j\delta t} [-\delta j \hat{\phi}_j(w) + c(w\hat{\phi}_j(w) - \phi_j(0)) - \rho[1-\hat{z}(w)]\hat{\phi}_j(w) + \rho[1-\hat{z}(w)]\hat{\phi}_{j-1}(w)] = 0.$$

So, for any $j = 0, 1, \ldots,$

$$-\delta j \hat{\phi}_j(w) + c(w\hat{\phi}_j(w) - \phi_j(0)) - \rho[1-\hat{z}(w)]\hat{\phi}_j(w) + \rho[1-\hat{z}(w)]\hat{\phi}_{j-1}(w) = 0.$$

Hence, we have

$$\hat{\phi}_j(w) = \frac{c\phi_j(0) - \rho[1-\hat{z}(w)]\hat{\phi}_{j-1}(w)}{cw - \rho[1-\hat{z}(w)] - \delta j}, \qquad j = 0, 1, \ldots.$$

For the initial case $j = 0$, note that $\hat{\phi}_{-1}(w) = 0$, we have

$$\hat{\phi}_0(w) = \frac{c\phi_0(0)}{cw - \rho[1-\hat{z}(w)]}.$$

By the boundary condition

$$\lim_{w\to 0} w\hat{\phi}_0(w) = \lim_{x\to\infty} \phi_0(x) = 1,$$

we have

$$\lim_{w\to 0} w\hat{\phi}_0(w) = \lim_{w\to 0} \frac{c\phi_0(0)}{c - \rho(1-\hat{z}(w))/w} = \frac{c\phi_0(0)}{c - \rho\mu_{1_Z}} = 1.$$

Then $\phi_0(0) = 1 - \rho\mu_{1_Z}/c$, and $\hat{\phi}_0(w)$ is as given by (20). Since $\hat{\phi}_j(w)$ exists at $w = W_j^+$ for any $j = 1, 2, \ldots,$ we have

$$\lim_{w\to W_j^+} (c\phi_j(0) - \rho[1-\hat{z}(w)]\hat{\phi}_{j-1}(w)) = 0,$$

and $\phi_j(0) = \rho c^{-1}[1-\hat{z}(W_j^+)]\hat{\phi}_{j-1}(W_j^+)$, $j = 1, 2, \ldots$. Hence, we have the recurrence relation between $\hat{\phi}_j(w)$ and $\hat{\phi}_{j-1}(w)$ as given in (19).

Remark 3. Theorem 2 will be used to derive a general asymptotic formula (given in Theorem 4 below), whereas Theorem 3 is more useful for obtaining an exact expression in the case of exponentially distributed claim sizes (given in Theorem 5 below).

4.2. Asymptotics of the ruin probability

Theorem 4. *Assume that $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$. Then the asymptotics of the ruin probability are given by*

$$\psi(x,t) \sim \sum_{j=0}^{\infty} \kappa_j(t)\mathrm{e}^{-R_j x} \quad \text{as } x \to \infty, \tag{21}$$

where

$$\kappa_0(t) := \mathrm{e}^{-cR_0\vartheta\mathrm{e}^{-\delta t}/\rho}\frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty z\mathrm{e}^{R_0 z}\,\mathrm{d}Z(z)-c},$$

$$\kappa_j(t) := \mathrm{e}^{-j\delta t}\frac{c\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(-R_j)]}}{\rho\int_0^\infty z\mathrm{e}^{R_j z}\,\mathrm{d}Z(z)-c}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}, \qquad j=1,2,\ldots.$$

Proof. Define $\phi(x,t) := \sum_{j=0}^{\infty}\phi_j(x,t)$. Then $\hat{\phi}(w,t) = \sum_{j=0}^{\infty}\hat{\phi}_j(w,t)$, where every $\hat{\phi}_j(w,t)$ term is specified by (9), i.e.

$$\hat{\phi}_0(w,t) := \mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\frac{c-\rho\mu_{1_Z}}{cw-\rho[1-\hat{z}(w)]}, \tag{22}$$

$$\hat{\phi}_j(w,t) := c\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\mathrm{e}^{-j\delta t}\frac{\sum_{\ell=0}^{j} r_\ell[\vartheta\hat{z}(w)]^{j-\ell}/(j-\ell)!}{cw-\rho[1-\hat{z}(w)]-\delta j}, \qquad j=1,2,\ldots. \tag{23}$$

Now we discuss the asymptotics of the terms $\phi_0(x,t)$ and $\{\phi_j(x,t)\}_{j=1,2,\ldots}$, respectively.

For $\phi_0(x,t)$, we have the asymptotics $1-\phi_0(x,t) \sim \kappa_0(t)\mathrm{e}^{-R_0 x}$ as $x \to \infty$, since, by the final value theorem,

$$\begin{aligned}
\kappa_0(t) &= \lim_{x\to\infty}\mathrm{e}^{R_0 x}(1-\phi_0(x,t)) \\
&= \lim_{w\to 0} w\mathcal{L}_w\{\mathrm{e}^{R_0 x}(1-\phi_0(x,t))\} \\
&= -\lim_{w\to 0} w\hat{\phi}_0(w-R_0,t) \\
&= -\lim_{w\to 0} w\frac{\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(w-R_0)]}(c-\rho\mu_{1_Z})}{c(w-R_0)-\rho[1-\hat{z}(w-R_0)]} \\
&= \frac{\mathrm{e}^{-cR_0\vartheta\mathrm{e}^{-\delta t}/\rho}(c-\rho\mu_{1_Z})}{\rho\int_0^\infty z\mathrm{e}^{R_0 z}\,\mathrm{d}Z(z)-c}.
\end{aligned}$$

For $\phi_j(x,t)$, $j=1,2,\ldots$, we have the asymptotics $-\phi_j(x,t) \sim \kappa_j(t)\mathrm{e}^{-R_j x}$ as $x \to \infty$, since, by the final value theorem,

$$\begin{aligned}
\kappa_j(t) &= \lim_{x\to\infty}\mathrm{e}^{R_j x}(-\phi_j(x,t)) \\
&= -\lim_{w\to 0} w\hat{\phi}_j(w-R_j,t) \\
&= -\lim_{w\to 0}\left(w\frac{c\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(w-R_j)]}\mathrm{e}^{-j\delta t}}{c(w-R_j)-\rho[1-\hat{z}(w-R_j)]-\delta j}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w-R_j)]^{j-\ell}}{(j-\ell)!}\right) \\
&= \frac{c\mathrm{e}^{\vartheta\mathrm{e}^{-\delta t}[1-\hat{z}(-R_j)]}\mathrm{e}^{-j\delta t}}{\rho\int_0^\infty z\mathrm{e}^{R_j z}\,\mathrm{d}Z(z)-c}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}.
\end{aligned}$$

Therefore, $\psi(x,t) = 1 - \phi(x,t) = 1 - \phi_0(x,t) + \sum_{j=1}^{\infty} -\phi_j(x,t)$, and (21) follows immediately.

Remark 4. Set $L(t) = 1 - \vartheta\delta \mathrm{e}^{-\delta t}/\rho$ and $t = 0$ in Theorem 1. Then $\int_0^\infty \bar{L}(s)\,\mathrm{d}s = \vartheta/\rho$ and we recover $\kappa_0(t)\mathrm{e}^{-R_0 x}$, the first-order asymptotics of the ruin probability obtained by Theorem 4. The higher-order asymptotics depend on the distributional property of the general distribution function L.

Remark 5. We can rewrite $\hat{\phi}_0(w,t)$ in (22) as

$$\begin{aligned}\hat{\phi}_0(w,t) &= \mathrm{e}^{\vartheta \mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\frac{p_0}{1-(1-p_0)(1-\hat{z}(w))/\mu_{1_Z} w} \\ &= \mathrm{e}^{\vartheta \mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\sum_{i=0}^{\infty} p_0(1-p_0)^i\left(\frac{1-\hat{z}(w)}{\mu_{1_Z} w}\right)^i, \qquad p_0 = 1 - \frac{\rho\mu_{1_Z}}{c}.\end{aligned}$$

The third term of $\hat{\phi}_0(w,t)$ above is the Laplace transform of a compound geometric distribution $\sum_{i=0}^{\infty} p_0(1-p_0)^i d_0^{(i)}(x)$, where $d_0^{(i)}(x)$ is the i-fold convolution of a proper density function $d_0(x) =: \bar{Z}(x)/\mu_{1_Z}$, since $0 < p_0 < 1$ and

$$\int_0^\infty d_0(x)\,\mathrm{d}x = \mathcal{L}_w\{d_0(x)\}\big|_{w=0} = \lim_{w\to 0}\frac{1-\hat{z}(w)}{\mu_{1_Z} w} = \frac{1}{\mu_{1_Z}}\mu_{1_Z} = 1.$$

For $j = 1, 2, \ldots$, we can also rewrite $\hat{\phi}_j(w,t)$ in (23) as

$$\begin{aligned}\hat{\phi}_j(w,t) &= p_j\left(1-(1-p_j)\frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}\right)^{-1} \\ &\quad\times \frac{1}{p_j}\frac{\mathrm{e}^{\vartheta \mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\mathrm{e}^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \\ &= \sum_{i=0}^{\infty} p_j(1-p_j)^i\left(\frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}\right)^i \\ &\quad\times \frac{1}{p_j}\frac{\mathrm{e}^{\vartheta \mathrm{e}^{-\delta t}[1-\hat{z}(w)]}\mathrm{e}^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!},\end{aligned}$$

where $p_j = \delta j/cW_j^+$. The first term of $\hat{\phi}_j(w,t)$ above is the Laplace transform of a compound geometric distribution $\sum_{i=0}^{\infty} p_j(1-p_j)^i d_j^{(i)}(x)$, where $d_j^{(i)}(x)$ is the i-fold convolution of a proper density function

$$d_j(x) =: \frac{W_j^+}{1-\hat{z}(W_j^+)}\mathrm{e}^{W_j^+ x}\int_x^\infty \mathrm{e}^{-W_j^+ z}\,\mathrm{d}Z(z),$$

since

$$0 < p_j = 1 - \frac{\rho}{c}\frac{1-\hat{z}(W_j^+)}{W_j^+} = \frac{\delta j}{cW_j^+} = \frac{\delta j}{\rho[1-\hat{z}(W_j^+)]+\delta j} < 1,$$

$$\int_0^\infty d_j(x)\,\mathrm{d}x = \mathcal{L}_w\{d_j(x)\}\big|_{w=0} = \frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}\bigg|_{w=0} = 1.$$

Note that, for a constant ν, we have $\mathcal{L}_w\{\mathrm{e}^{\nu x}\int_x^\infty \mathrm{e}^{-\nu z}\,\mathrm{d}Z(z)\} = (\hat{z}(\nu)-\hat{z}(w))/(w-\nu)$, which is a special case of the double Dickson–Hipp operator introduced in Dickson and Hipp (2001).

5. Ruin with exponentially delayed claims and exponentially distributed sizes

The asymptotic formula in (21) becomes exact if the claim sizes follow an exponential distribution.

Theorem 5. *Assume that $c > \rho\mu_{1_Z}$, $L \sim \mathrm{Exp}(\delta)$, and that Z follows an exponential distribution. Then the ruin probability is given by*

$$\psi(x,t) = \sum_{j=0}^{\infty} \kappa_j(t)\mathrm{e}^{-R_j x}. \tag{24}$$

Proof. By Theorem 3, if $Z \sim \mathrm{Exp}(\gamma)$ then, for $j = 0$, we have

$$\hat{\phi}_0(w) = \frac{c-\rho/\gamma}{cw-\rho w/(\gamma+w)} = \left(1-\frac{\rho}{c\gamma}\right)\frac{\gamma+w}{(w+R_0)w}. \tag{25}$$

For $j = 1, 2, \ldots$, we have

$$\begin{aligned}\hat{\phi}_j(w) &= \rho\frac{W_j^+\hat{\phi}_{j-1}(W_j^+)/(\gamma+W_j^+) - w\hat{\phi}_{j-1}(w)/(\gamma+w)}{cw-\rho w/(\gamma+w)-\delta j}\\ &= \rho\frac{(W_j^+\hat{\phi}_{j-1}(W_j^+) - w\hat{\phi}_{j-1}(w))/(w-W_j^+) + W_j^+\hat{\phi}_{j-1}(W_j^+)/(\gamma+W_j^+)}{c(w+R_j)}.\end{aligned}$$

In particular, for $j = 1$, we observe that

$$\begin{aligned}\hat{\phi}_1(w) &= \rho\frac{(W_1^+\hat{\phi}_0(W_1^+) - w\hat{\phi}_0(w))/(w-W_1^+) + W_1^+\hat{\phi}_0(W_1^+)/(\gamma+W_1^+)}{c(w+R_1)}\\ &= \rho\frac{(1-\rho/c\gamma)(\gamma-R_0)/(W_1^++R_0) + W_1^+\hat{\phi}_0(W_1^+)(w+R_0)/(\gamma+W_1^+)}{c(w+R_0)(w+R_1)},\end{aligned}$$

which is the Laplace transform of a linear combination of e^{-R_0x} and e^{-R_1x}. In general, for $j = 1, 2, \ldots$, assume that

$$\hat{\phi}_j(w) = \frac{P_j(w)}{c\prod_{i=0}^{j}(w+R_i)}, \qquad j = 1, 2, \ldots,$$

where $\{P_j(w)\}_{j=1,2,\ldots}$ are functions of w. Then

$$\begin{aligned}P_j(w) = \frac{\rho}{c}\prod_{i=0}^{j-1}(w+R_i)\Bigg[&\frac{W_j^+P_{j-1}(W_j^+)/\prod_{i=0}^{j-1}(W_j^++R_i) - wP_{j-1}(w)/\prod_{i=0}^{j-1}(w+R_i)}{w-W_j^+}\\ &+\frac{W_j^+}{\gamma+W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^++R_i)}\Bigg],\end{aligned}$$

and we have

$$P_j(w) = \frac{\rho}{c}\Bigg[\frac{W_j^+}{w - W_j^+}\Bigg(\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)}\prod_{i=0}^{j-1}(w + R_i) - wP_{j-1}(w)\Bigg)$$
$$+ \frac{W_j^+}{\gamma + W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)}\prod_{i=0}^{j-1}(w + R_i)\Bigg], \qquad j = 2, 3, \ldots,$$
$$P_1(w) = \rho\Bigg[\Bigg(1 - \frac{\rho}{c\gamma}\Bigg)\frac{\gamma - R_0}{W_1^+ + R_0} + \frac{W_1^+}{\gamma + W_1^+}\hat{\phi}_0(W_1^+)(w + R_0)\Bigg].$$

Note that, for $j = 2, 3, \ldots$, $w = W_j^+$ is one of the roots of the numerator of the first term, so the denominator $w - W_j^+$ then cancels. The function $P_1(w)$ is a polynomial of degree 1, and, obviously, by the method of induction, $\{P_j(w)\}_{j=1,2,\ldots}$ are polynomial functions of w with maximum degree j. Hence, for any $j = 1, 2, \ldots$, we have the partial fraction decomposition

$$\frac{P_j(w)}{c\prod_{i=0}^{j}(w + R_i)} = \sum_{i=0}^{j} b_{ji}\frac{1}{w + R_i},$$

where $\{b_{ji}\}_{i=0,1,\ldots,j}$ are all constants. Since $\mathcal{L}_w\{\mathrm{e}^{-R_i x}\} = 1/(w + R_i)$, $i = 0, 1, \ldots, j$, we have $\phi_j(x) = \sum_{i=0}^{j} b_{ji}\mathrm{e}^{-R_i x}$, $j = 1, 2, \ldots$. For $j = 0$, we have $R_0 = \gamma - \rho/c$, and rewrite (25) as

$$\hat{\phi}_0(w) = \frac{1}{w} - \frac{\rho}{c\gamma}\frac{1}{w + R_0},$$

which is the Laplace transform of $\phi_0(x) = 1 - \rho(c\gamma)^{-1}\mathrm{e}^{-R_0 x}$. Then the ruin probability $\psi(x, t)$ is a linear combination of $\{\mathrm{e}^{-R_j x}\}_{j=0,1,\ldots}$, since

$$\begin{aligned}\psi(x, t) &= 1 - \phi(x, t)\\ &= 1 - \phi_0(x) - \sum_{j=1}^{\infty}\mathrm{e}^{-j\delta t}\phi_j(x)\\ &= \frac{\rho}{c\gamma}\mathrm{e}^{-R_0 x} - \sum_{j=1}^{\infty}\mathrm{e}^{-j\delta t}\sum_{i=0}^{j} b_{ji}\mathrm{e}^{-R_i x}\\ &= \sum_{j=0}^{\infty} B_j(t)\mathrm{e}^{-R_j x},\end{aligned}$$

where $\{B_j(t)\}_{j=0,1,\ldots}$ are all deterministic functions of time t. Then (5) should hold, because the asymptotic representation given by Theorem 4 is also a linear combination of $\{\mathrm{e}^{-R_j x}\}_{j=0,1,\ldots}$.

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